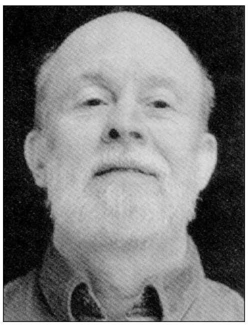
A Student Project on Wind Chimes

Tuning in to Standing Waves

G. William Baxter and Keith M. Hagenbuch



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Every introductory physics class includes a discussion of standing waves. We wanted to offer our students an opportunity to earn bonus points by demonstrating their knowledge of this material. The project we developed involves constructing a set of wind chimes in which each hanging rod produces a clear tone, and the combination of rods generates a chord. The project is not concerned with exact pitches, but rather with standing waves and nodes and the correct relationship between the pitches in the chord. Our students have enjoyed this project while learning something from it, and we thought that other physics teachers might want to consider the idea for use with their classes. This project can be done at many different levels, high school through college, depending on the mathematical knowledge of the class.

Although wind chimes are commonly available and easily constructed, there is little in the physics literature about their design.[1,2] Discussions of the oscillation modes of free rods are more common in engineering literature;[3] "artistic" wind chimes appear to be constructed chiefly by trial and error.

Getting Started

We suggest that you provide the following instructions and information to students who are interested in a wind-chime project.

- The wind-chime rods are to produce a chord—either "barbershop 7th," a C9th, or a Cllth. (For helpful information, see the Oliver note on page 209, or refer to an introductory text on the physics of music.[4])
- Leave choice of materials to the students, but provide a table of Young's moduli for common materials[5] and Eq. (6) from this paper, with $(kL)_1 = 4.73044$, so that they can compute the expected frequencies.
- Provide a table showing the frequencies of notes on the chromatic scale and the relationship between notes. (See Oliver note on page 209.)
- Have available a computer with a microphone and an A-to-D converter so students can determine the actual frequencies of each chime.
- Tell the students that the fundamental has a node 22.4% of the length of the rod from the end.

This node location differs slightly from 25% that would be expected for a simple sinusoidal amplitude such as those typically discussed in a course of this type; this is the first hint to students that class discussion of waves on strings may not contain the whole story. This project introduces students to waves that do not satisfy $\lambda f = v$.

Upper-level students should be given the opportunity to solve the fourth-order wave equation, but this is not necessary for introductory students. Most commonly, problems arise when students do not hang the rods at their nodes or do not strike them at an antinode. Introductory students should be encouraged to hear the difference between rods hung at nodes and rods not hung at nodes.

Theory

Our introductory physics courses concentrate on waves that are described by solutions to a second-order wave equation

$$\frac{\partial^2 y}{\partial t^2} - v^2 \frac{\partial^2 y}{\partial x^2} = 0 \qquad (1)$$

such as waves on strings or sound waves in air. They do not generally consider other wave equations. Here, the equation describing the motion of a free rod is fourth order,[3]

$$\frac{\partial^2 y}{\partial t^2} + \frac{EI}{A\rho}\frac{\partial^4 y}{\partial x^4} = 0 \qquad (2)$$

where A is the cross-sectional area of the rod, E is Young's modulus for the material, ρ is the density of the material, and I is the moment of the area.

The solution of Eq. (2), which is given in the Appendix, has the form $y(x, t) = X(x)\, e^{\pm i\omega t}$. The amplitude of standing waves on the rod $X(x)$ has the form

$$X(x) = A \left\{ \cos(kx) + \cosh(kx) + \left[\frac{\cosh(kL) - \cos(kL)}{\sin(kL) - \sinh(kL)} \right] \times [\sin(kx) + \sinh(kx)] \right\} \quad (3)$$

where only the k's satisfying

$$\cos(kL)\cosh(kL) = 1 \quad (4)$$

are allowed. Solutions are the crossing points in Fig. 1. Here we have actually plotted $[\cos(kL)\cosh(kL) - 1]/\cosh(kL)$. The numerator determines the zero crossing points. (The denominator has been added to keep the plotted quantity from getting too large.)

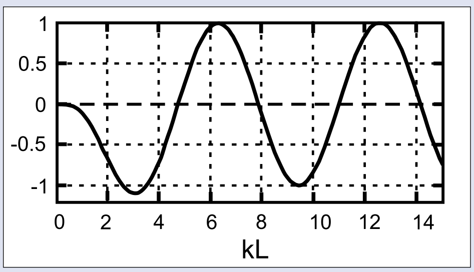


Fig. 1. Zeroes are the solutions to Eq. (4).

Zero	kL	First Node
0	0.00	
1	4.73004	0.224L
2	7.8532	0.1321L
3	10.995601	0.0944L

Table I.

Note that the 0 solution is not allowed since it causes the coefficient of the third term in Eq. (3) to be undefined. Solutions are plotted in Fig. 2 for the first three modes of oscillation. Notice that the nodal points for the first, second, and third modes are at approximately 22.4%, 13%, and 9% of the rod length, respectively, from one end.

Discussion

Construction of a set of wind chimes requires a combination of thought, creativity, and experimentation. Our students used copper, steel, and galvanized steel stock, both solid rod and tubing. Most were hung at the recommended location, but some were suspended at the center and some at the end. The latter rang poorly.

Indeed, the rod not suspended exactly from the node point damped out much more quickly when rung. Moreover, several students noted that it was important that the hanging wire not contact the rod anywhere other than the node point, and they came up with some creative hanging mechanisms to prevent this. A few students noted that the rods should be hung so that they will be struck only at the antinode in order to excite the correct standing wave at the initial impact. Thus, even without understanding the intricacies of this particular system, the students were able to experimentally explore several features of standing waves.

Note that the standing waves shown in Fig. 2 do bear a resemblance to the sinusoidal waves that we expect in the case of the second-order wave equation, though some of the nodes are shifted. There are some notable differences. For example, the longest standing wave has two nodes rather than one. A standing wave with one node at the center of the rod and an antinode at each end would correspond to a rotation of the entire rod, though not a pure rotation. This case is not possible with a free rod but only with a rod clamped at its center.

Determination of the actual frequencies in this system is somewhat different from that for waves in systems described by second-order wave equations. Although most introductory physics texts discuss only second-order wave equations, vibrating rods are sometimes used as an example. Here, Eq. (2) suggests that the standing waves are

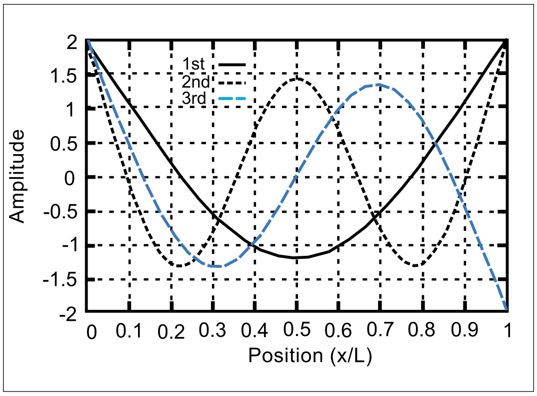
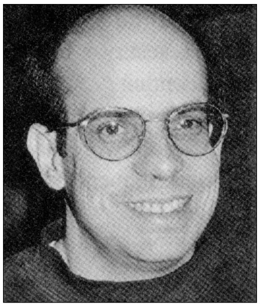


Fig. 2. Standing waves on a free rod as given by Eq. (3).



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described by

$$\omega = k^2 \sqrt{\frac{EI}{A\rho}} \tag{5}$$

or

$$f_m = \frac{k_m^2}{2\pi}\sqrt{\frac{EI}{A\rho}} = \frac{(k\,L)_m^2}{2\pi L^2}\sqrt{\frac{EI}{A\rho}} \tag{6}$$

where $(k\,L)_m$ is one of the zero crossing points from Table I. For hollow tubing, $I/A = (r_1^2 + r_2^2)/4$, where r_1 and r_2 are the inner and outer radii, respectively. Thus, the frequency is proportional to the inverse square of the length of the rod. We have been able to test this by using a computer with a microphone to examine the power spectra, such as Fig. 3, of rods of different lengths suspended at the 22.4% point. These test rods were constructed of galvanized steel electrical conduit (OD 23.4 mm, ID 21.0 mm, $E = 200 \times 10^9$ Pa, and $\rho = 7800$ kg/m^3). The resulting fundamentals are given in the following table where $f_{predicted}$ is the theoretically predicted frequency, and f_{actual} is the experimentally measured frequency.

Length (cm)	$f_{predicted}$(Hz)	f_{actual} (Hz)
22.0	2930	2775
24.1	2440	2340
25.8	2130	2040
28.0	1810	1730
29.3	1650	1580
32.9	1310	1280
35.9	1100	1080
38.0	983	975
42.0	805	804
59.9	396	390

The spectrum changes dramatically when the rod is not hung at a node. Figure 4 demonstrates the increased noise present when a rod is supported at the end rather than at a node.

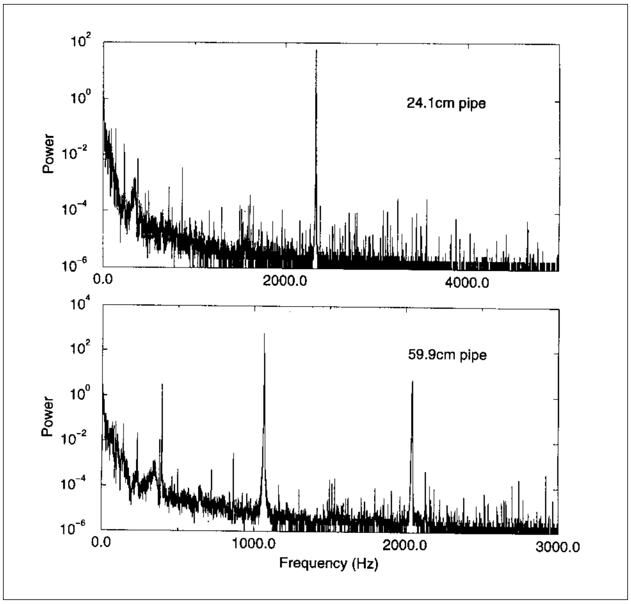

Fig. 3. Power spectra of two tubes hung at the 22.4% node point. For the 59.9-cm tube, the three peaks correspond to the first (396 Hz), second (1091 Hz), and third (2139 Hz) harmonics.

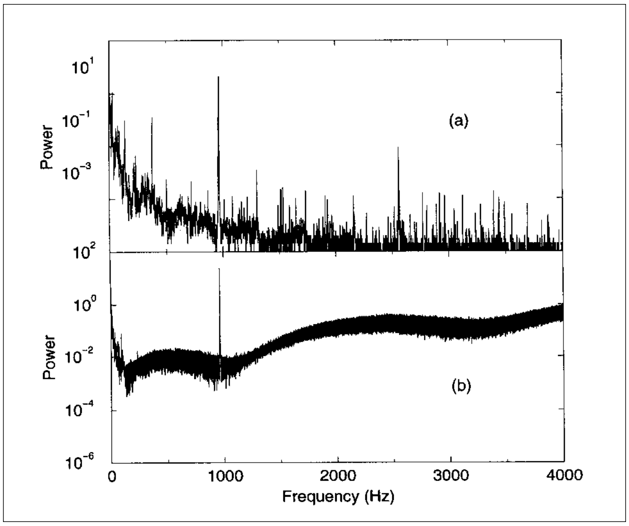

Fig. 4. (a) Power spectra of a 38.0-cm tube hung at the 22.4% node point. (b) Power spectra of a 38.0-cm tube hung at the end. Note the increased background noise level when the rod is not hung at a node point even though the fundamental is still present.

Conclusion

Wind-chime construction can be an excellent student project to explore standing waves. Metal rods and tubes sound grand and ring well when hung exactly at the nodes, whereas anything else leads to damping and excitation of other modes, each of which is unpleasant to the human ear. It is also possible to hear other modes being excited when a properly hung rod is not struck at an antinode (as in Fig. 3).

To correctly calculate frequencies, it is necessary to recognize that this system is not described by a second-order wave equation. This becomes another example of how physics is done: study the simple case first, and then add on levels of complexity. The math here is accessible to students who have completed an undergraduate differential-equation course.

The educational merit of this project is great, but there may be an even better reason for trying this project. It is extremely satisfying to design and build a wind chime that is equal or better in sound quality to the most expensive commercial wind chimes. The students' results sound great!

Acknowledgments

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Appendix: The Rest of the Story...

While we often have to deal with incomplete information, physicists prefer to know as much as possible about every problem. In this case, we can motivate the form of the fourth-order wave equation.

$$A\rho\frac{\partial^2 y}{\partial t^2} + EI\frac{\partial^4 y}{\partial x^4} = 0 \qquad (7)$$

where A is the cross-sectional area of the rod, E is Young's modulus for the material, ρ is the density of the material, and I is the moment of the area; y is the displacement of the center of the rod from equilibrium and x is the distance along the rod.

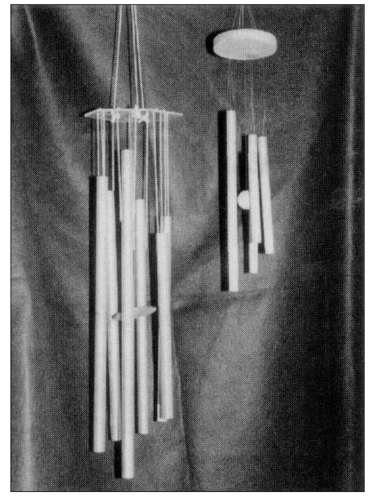
In deriving the wave equation for a string, Eq. (1), we begin with a segment under tensile stress. To produce a restoring force, the segment must have a slope that is different at one end than at the other.[6] The restoring force is thus proportional to the curvature of the segment. So, once the system is under tension, the second derivative with respect to length is required to produce a restoring force.

The situation is different for a free rod. There is no tensile stress unless the rod is curved. (A free rod that is rotated as a rigid body experiences no restoring force.) For the rod, curvature produces stress and two derivatives beyond this are needed to produce a restoring force, as was the case with string. The cases of string and rod are both shown in Fig. 5. Young's modulus, E, is defined as stress per strain and has units of force per unit area. Here strain is proportional to the curvature, $\partial^2 y/\partial x^2$, at a point along the rod. So, $EA\partial^2 y/\partial x^2$ is proportional to stress times cross-sectional area or force. Using the moment of area, defined as $I = \int h^2 \mathrm{d}A$, where the integral is over the entire cross section, allows elements farther from the neutral line to have a greater contribution. Therefore, $EI\, \partial^2 y/\partial x^2$ is proportional to force and is properly weighted for contributions of area elements farther from the "neutral" line at a point x. For a cross section of the rod $A\mathrm{d}x$, the net force is the difference between the forces at x and $x + \mathrm{d}x$. By analogy with the string problem, this is the second derivative. Therefore, $-EI\partial^4 y/\partial x^4$ is the net force per unit length on the rod cross section located at x. Then, applying Newton's second law to a piece of the rod yields

$$\sum F/L = -EI\frac{\partial^4 y}{\partial x^4} = \frac{\mathrm{m}a}{L} = A\rho\frac{\partial^2 y}{\partial t^2} \qquad (8)$$

which is Eq. (2). In any physics calculation, it is always a good idea to check the dimensions. Here, both $EI\, \partial^4 y/\partial x^4$ and $A\rho\, \partial^2 y/\partial t^2$ have units of N/m or force per unit length.

Equations that vary in space and time are typically solved by the method of separation of variables. We assume a solution of the form $y(x,t) = X(x)e^{\pm i\omega t}$. Substituting this into Eq. (2) results in a time-independent equation for the amplitude of oscillation of points along the length of the rod.



Student-constructed wind chimes.

$$\frac{\mathrm{d}^4 X}{\mathrm{d}x^4} - \omega^2\frac{A\rho}{EI}X = 0 \qquad (9)$$

Once we have this equation, we can solve it using the time-honored physics tradition of guessing a solution and applying boundary conditions. It is quite easy to pick unique functions that produce the same function when differentiated four times: cos ( ), sin ( ), cosh ( ), and sinh ( ). [Note that exp ( ) also works, but it is already a combina-

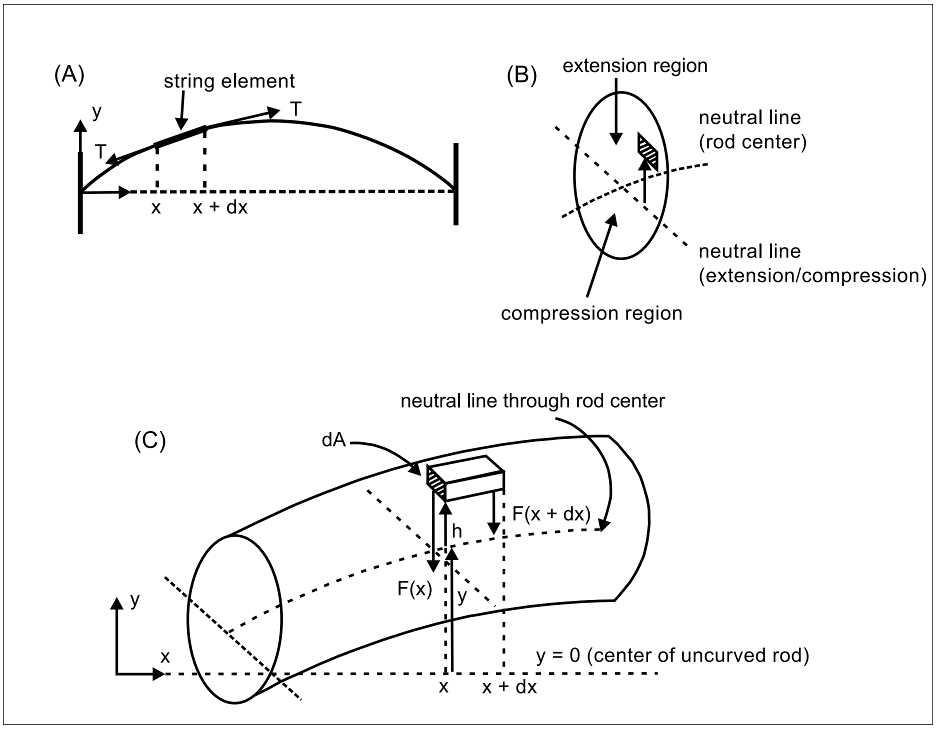


Fig. 5. (A) Forces on an element of a stretched string fixed at each end: T = tension. (B) Cross section of rod showing calculation of moment of area. Note that *h* is the perpendicular to the neutral line, which separates the region where the material is being compressed from the region of material being extended or stretched. (C) Geometry of the curved rod.

tion of two of the previous four functions.] A general solution that is a linear combination of these is

$$X(x) = A\cos(kx) + B\sin(kx) + C\cosh(kx) + D\sinh(kx) \tag{10}$$

The argument is kx in each case with $k = 2\pi/\lambda$ because this describes the shape of the rod at a particular time and the rod must select a single solution. Since the rod is free at each end, we expect no bending and no shear at the ends. (With no material beyond the end, there is nothing to provide a restoring force so there is no second or third derivative.)

$$\frac{d^2X}{dx^2} = \frac{d^3X}{dx^3} = 0 \tag{11}$$

Applying these boundary conditions at the ends of the rod, $x = 0$ and $x = L$, produces two equations for each end. Applying the boundary conditions at $x = 0$,

$$-k^2B + k^2D = 0 \tag{12}$$
$$-k^3A + k^3C = 0 \tag{13}$$

It is clear that $A = C$ and $B = D$. Applying the boundary conditions at $x = L$,

$$-k^2A\cos(kL) - k^2B\sin(kL) + k^2C\cosh(kL) + k^2D\sinh(kL) = 0 \tag{14}$$

$$+k^3A\sin(kL) - k^3B\cos(kL) + k^3C\cosh(kL) + k^3D\sinh(kL) = 0 \tag{15}$$

Substituting for C and D, and solving for B,

$$B = A\left[\frac{\cosh(kL) - \cos(kL)}{\sin(kL) - \sinh(kL)}\right] \tag{16}$$

Eliminating A and B, yields

$$[\sin(kL) - \sinh(kL)]\,[\sin(kL) + \sinh(kL)] = -[\cosh(kL) - \cos(kL)]^2 \tag{17}$$

which can be simplified using trigonometric relations to

$$\cos(kL)\cosh(kL) = 1 \tag{18}$$

This is the condition, previously Eq. (4), that the k's must satisfy. With this information, the amplitude of standing waves on the rod becomes

$$X(x) = A\left\{\cos(kx) + \cosh(kx) + \left[\frac{\cosh(kL) - \cos(kL)}{\sin(kL) - \sinh(kL)}\right] \times [\sin(kx) + \sinh(kx)]\right\} \tag{19}$$

where only the k's satisfying Eq. (4) are allowed.

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